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A SERIES USEFUL IN THE STUDY OF DIFFERENCE EQUATIONS

by

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The series treated in this paper had its origin in the study of
difference equations with varying difference interval. It is hoped
to follow in a short time with a paper on this subject.

We shall be concerned with series of the type

$$(1) \qquad \sum_{n=1}^{\infty} c_n \left[(z + h_1(z))(z + h_2(z)) \ldots (z + h_n(z)) \right]^{-1}.$$

Here $z = x + yi$ with x and y real. We shall require in addition that
$0 < \epsilon \leq h_n(z) \leq E(n)$, that $h_n(z) \to \infty$ when $n \to \infty$ and that
$x + h_n(z)$ increase with x.

1. Convergence Theorems.

We now give a lemma which is well known. However, the proof is
so short that it is given as adding to the understanding of the work
which follows.

Let $\triangle_j b_j(z) = b_{j+1}(z) - b_j(z)$:

Lemma I: **Hypotheses;** (1) $a_n(z)$ **and** $b_n(z)$, **for all** n, **are defined at all points of a region** , R, **of the complex plane.** (2) $\sum\limits_{n=1}^{\infty} a_n(z)$ **converges uniformly over** R. (3) $b_n(z)$ **is uniformly bounded over** R. (4) $\sum\limits_{j=m}^{m'} \left| \triangle_j b_j(z) \right|$ **is uniformly bounded over** R. **Conclusion:** $\sum\limits_{n=1}^{\infty} a_n(z) \, b_n(z)$ **converges uniformly over** R.

Proof: Using summation by parts

$$\sum_{n=m}^{m'} a_n(z) \, b_n(z) = b_n(z) \sum_{j=m}^{n-1} a_j \Bigg]_m^{m'+1} - \sum_{n=m}^{m'} \triangle_n b_n(z) \sum_{j=m}^{n} a_j$$

$$\left| \sum_{n=m}^{m'} a_n(z) \, b_n(z) \right| \leq \left| b_{m'+1}(z) \right| \left| \sum_{j=m}^{m'} a_j \right| + \sum_{n=m}^{m'} \left| \triangle_n b_n(z) \right| \left| \sum_{j=m}^{n} a_j \right| .$$

$$\leqq M \, \varepsilon + N \, \varepsilon < \eta \qquad \text{which proves the lemma.}$$

Let

$$(2) \quad b_n(z) = \frac{(z_0 + h_1(z_0))(z_0 + h_2(z_0))\dots(z_0 + h_n(z_0))}{(z + h_1(z))(z + h_2(z))\dots(z + h_n(z))}$$

This expression plays an important role in the discussions that follow.

Let

$$(3) \quad b_n(x) = \frac{(x_0 + h_1(z_0))(x_0 + h_2(z_0))\dots(x_0 + h_n(z_0))}{(x + h_1(z))(x + h_2(z))\dots(x + h_n(z))}$$

Theorem I: _If_ $\sum_{n=1}^{\infty} [h_n(z_0)]^{-2}$ _converges_ _and_ $0 \leq x_0 \leq x$ _then given_

any _positive numbers,_ _a_ _and_ _b,_ _and_ _a positive integer,_ _p,_ _there exists_

_an_ _N_ _such that_

(4) $\quad \left| b_n(z) \right| < N/(a(x - x_0)^p + b), \quad n > p.$

Proof:

(5) $\quad \left| b_n(z) \right| = \dfrac{\left| z_0 + h_1(z_0) \right| \cdot \left| z_0 + h_2(z_0) \right| \cdot \ldots \cdot \left| z_0 + h_n(z_0) \right|}{(x_0 + h_1(z_0))(x_0 + h_2(z_0))\ldots(x_0 + h_n(z_0))} \cdot$

$\dfrac{(x + h_1(z))(x + h_2(z))\ldots(x + h_n(z))}{\left| z + h_1(z) \right| \cdot \left| z + h_2(z) \right| \cdot \ldots \cdot \left| z + h_n(z) \right|} \cdot b_n(x) \leq$

$\left\{ \left[1 + \left(\dfrac{y_0}{x_0 + h_1(z_0)}\right)^2\right]\left[1 + \left(\dfrac{y_0}{x_0 + h_2(z_0)}\right)^2\right]\ldots\left[1 + \left(\dfrac{y_0}{x_0 + h_n(z_0)}\right)^2\right] \right\}^{1/2} b_n(x)$

$< \bar{N} b_n(x).$

This follows from the fact that $\dfrac{(x + h_1(z))\ldots(x + h_n(z))}{\left| z + h_1(z) \right| \ldots \left| z + h_n(z) \right|} \leq 1,$

and the fact that $\sum_{n=1}^{\infty} [h_n(z_0)]^{-2}$ converges.

Hence

$\left| b_n(z) \right| \leq \dfrac{\bar{N}(x_0 + h_1(z_0))\ldots(x_0 + h_p(z_0))}{(x + h_1(z))\ldots(x + h_p(z))}$

since $\dfrac{(x_0 + h_{p+1}(z_0))\ldots(x_0 + h_n(z_0))}{(x + h_{p+1}(z))\ldots(x + h_n(z))} \leq 1$

Consequently

$$\left| b_n(z) \right| \leqq \frac{C \, \overline{N}(x_o + h_1(z_o))\ldots(x_o + h_p(z_o))}{C \, [x^p + h_1(z)h_2(z)\ldots h_p(z)]} < \frac{N}{a(x-x_o)^p + b}$$

provided $N > C \, \overline{N}(x_o + h_1(z_o))\ldots(x_o + h_p(x_o))$, $c > a$ and

$C \, h_1(z)h_2(z)\ldots h_p(z) > b$. It is always possible so to choose C since

p is fixed and $h_n(z) \geqq \varepsilon > 0$.

We have made the hypothesis that $n > p$ which is trivial in
convergence problems.

Theorem II: $b_n(z)$ is uniformly bounded in the half plane, $x \geqq 0$

This was proved incidentally in the proof of the last theorem
since $0 < b_n(x) \leqq 1$

Lemma II: $\dfrac{\left| z - z_o + h_{n+1}(z) - h_{n+1}(z_o) \right|}{x - x_o + h_{n+1}(z) - h_{n+1}(z_o)} \leqq \dfrac{\left| z - z_o \right|}{x - x_o}$, $x > x_o$.

Proof:

$$(6) \quad \frac{\left| z - z_o + h_{n+1}(z) - h_{n+1}(z_o) \right|}{x - x_o + h_{n+1}(z) - h_{n+1}(z_o)} = \left[1 + \frac{\left(\dfrac{y - y_o}{x - x_o} \right)^2}{\left(1 + \left(\dfrac{h_{n+1}(z) - h_{n+1}(z_o)}{x - x_o} \right) \right)^2} \right]^{1/2}$$

$$\leqq \left[1 + \left(\frac{y - y_o}{x - x_o} \right)^2 \right]^{1/2} = \frac{\left| z - z_o \right|}{x - x_o}$$

Theorem III: _If_ $x \geq x_o \geq 0$, _and_ $h_{n+1}(z) - h_n(z_o) \geq 0$ _and if_

$$\sum_{n=1}^{\infty} \left(\frac{1}{h_n(z)} \right)^2 \text{ converges uniformly when } x \geq \bar{x}_o, \text{ then}$$

$$\sum_{n=m}^{m'} \left| \Delta\, b_n(z) \right| \text{ is uniformly bounded over the portion of the complex}$$

plane determined by $(y - y_o)/(x - x_o) \leq a(x - x_o)^p + b$. _Here_

$\underline{a}$ _and_ $\underline{b}$ _are any positive constants and_ $\underline{p}$ _any positive integer,_ $x > x_o$.

Proof: From the definition

$$\left| \underset{n}{\Delta}\, b_n(z) \right| = \frac{\left| z - z_o + h_{n+1}(z) - h_{n+1}(z_o) \right|}{\left| z + h_{n+1}(z) \right|} \left| b_n(z) \right|.$$

But by Theorem I

$$\left| b_n(z) \right| \leq N\, b_n(x)$$

Hence, using the previous lemma,

$$\left| \Delta\, b_n(z) \right| \leq N \frac{\left| z - z_o + h_{n+1}(z) - h_{n+1}(z_o) \right|}{x - x_o + h_{n+1}(z) - h_{n+1}(x_o)} \cdot \frac{x + h_{n+1}(z)}{\left| z + h_{n+1}(z) \right|} \cdot$$

$$\frac{x - x_o + h_{n+1}(z) - h_{n+1}(z_o)}{x + h_{n+1}(z)}\, b_n(x)$$

$$\leq - N \frac{\left| z - z_o \right|}{x - x_o}\, \Delta\, b_n(x)$$

From this

$$\sum_{n=m}^{m'} \left| \Delta\, b_n(z) \right| \leq - N \sqrt{1 + \left(\frac{y - y_o}{x - x_o} \right)^2} \left(\frac{b_{m'+1}(x) - b_m(x)}{2} \right)$$

$$\leq N \sqrt{1 + \left(\frac{y - y_o}{x - x_o} \right)^2}\, b_m(x) \leq M \frac{\sqrt{1 + \left(\frac{y - y_o}{x - x_o} \right)^2}}{a(x - x_o)^p + b}$$

$$\leq M \sqrt{\frac{1 + [a(x - x_0)^p + b]^2}{a(x - x_0)^p + b}} < \overline{M}$$

which completes the proof.

Lemma III: If $z_0 = 0$, $x > 0$ and $\dfrac{|y|}{x + h_n(z) - h_n(z_0)} < m$

then $\dfrac{|z + h_n(z) - h_n(z_0)|}{|z + h_n(z)| - h_n(0)} < M$, where m and M are constants.

Proof: We note that $|z + h_n(z)| - h_n(0) > 0$. This follows from the fact that $x + h_n(z) > 0 + h_n(0)$. This is true since $x + h_n(z)$ increases with x.

$$\left| z + h_n(z) \right| - h_n(0) \geq x + h_n(z) - h_n(0)$$

Hence

$$\frac{\left| z + h_n(z) - h_n(0) \right|}{\left| z + h_n(z) \right| - h_n(0)} \leq \frac{\left| z + h_n(z) - h_n(0) \right|}{x + h_n(z) - h_n(0)}$$

$$= \left\{ 1 + \left(\frac{y}{x + h_n(z) - h_n(0)} \right)^2 \right\}^{1/2} < \sqrt{1 + m^2} < M$$

The conditions just given, namely, $z_0 = 0$, $x > 0$, $|y|/(x + h_n(z) - h_n(0)) \leq m$, serve to define a region, S, as follows. Let us suppose that $\left| h_n(z) - h_n(0) \right|$ has an upper bound g then a sectorial region is defined by $|y| \leq m(x + g)$. If $|y| \leq m(x + h_n(z) - h_n(0))$ then z lies in S.

Theorem IV: $\Delta b_n(z)$ is uniformly bounded over the region, S, above defined.

Proof:

$$\left| \Delta b_n(z) \right| = \frac{\left| z + h_n(z) - h_n(0) \right|}{\left| z + h_n(z) \right| - \left| h_n(0) \right|} \left| b_n(z) \right|$$

Now $\left| b_n(z) \right| \leqq N$ and by Lemma III $\dfrac{\left| z + h_n(z) - h_n(0) \right|}{\left| z + h_n(z) \right| - \left| h_n(0) \right|} < M.$

The theorem follows.

Theorem V: If series (1) converges at z_0 and if $x_0 \geq 0$ and if $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{h_n(z)} \right)^{z}$ converges uniformly over a sectorial region, R, determined by $\left| y - y_0 \right| (x - x_0)^{-1} \leqq a(x - x_0)^P + b$; then (1) converges uniformly over R also.

Proof: This theorem is an immediate consequence of lemma I and theorem II and III.

Theorem VI: If (1) converges at 0 then it converges uniformly over the region, S, of lemma III.

Proof: Proof again is an immediate consequence of lemma I and theorem IV.

Theorem VII: If (1) converges absolutely at z_0, when $x_0 \geqq 0$ then it converges absolutely uniformly over the half plane determined by $x \geq x_0$.

Proof: This theorem follows from the fact that $z + h_n(z)$ is increasing in x.

Theorem VIII: If (1) converges at $z = z_0$, $x_0 \geq 0$ then it converges at all points of the half plane $x > x_0$.

The conditions on h in theorems V, VI, VII, and VIII are those in the lemmas and theorems referred to.

2. Uniqueness Theorems.

Theorem IX: If

$$(7) \quad \sum_{n=1}^{\infty} \frac{c_n}{(z + h_1(z))(z + h_2(z))\ldots(z + h_n(z))} \equiv 0$$

over its half-plane of convergence, $x \geq x_0 \geq 0$, then $c_n = 0$, $n = 1, 2, \ldots$

Proof: When x becomes infinite each term approaches zero. Convergence is uniform. We can assume the c's functions of $h(z)$ but add the hypothesis, $\dfrac{c_n}{z}$ approaches zero when x becomes infinite.

$$(8) \quad \frac{c_1}{z + h_1(z)} + \frac{c_2}{(z + h_1(z))(z + h_2(z))} + \frac{c_3}{(z+h_1(z))(z+h_2(z))(z+h_3(z))} + \ldots \equiv 0$$

From which

$$(9) \quad c_1 + \frac{c_2}{z + h_2(z)} + \frac{c_3}{(z + h_2(z))(z + h_3(z))} + \ldots \equiv 0$$

This is a series of the same type as (8). It converges uniformly

$x \geqq x_0$. Let x become infinite and we have $c_1 = 0$. We now begin

all over again in a familiar way and get $c_2 = 0$ and then

c_3, c_4, $\ldots c_n$, $\ldots$

Theorem X: $\Omega(z) \; \overline{\Omega}(z) = 0$ at an infinite number of points in the

finite half plane $x_0 \leqq x < M$ then $\Omega(z)$ and/or $\overline{\Omega}(z)$ is identically

zero.

Proof: Either $\Omega(z)$ or $\overline{\Omega}(z)$ must have an infinite number of

finite zeros. This is impossible for an analytic function unless

it is identically zero.

Part II.

In this part of the paper the independent variable is real. It is denoted by x. We call attention to the fact that $h_n(x)$ is real and positive and that $h_{n+1}(x) > h_n(x)$ and $x + h_n(x)$ increasing with x also that $x + h_n(x) > 0$

3. Step up Theorems.

We consider series (1) and assume that it is absolutely convergent for all values of $\cdot x$ considered.

Let

$$|c_n| = a_n$$

and let

$$(10) \quad \Omega_1(x) = \sum_{j=1}^{\infty} \frac{c_j}{(x + h_1(x))(x + h_2(x))\ldots(x + h_j(x))}$$

$$(11) \quad \Omega_2(x) = \sum_{j=1}^{\infty} \frac{a_j}{(x + h_1(x))(x + h_2(x))\ldots(x + h_j(x))}$$

We shall carry on certain operations on series (10) and (11). These operations are to be precisely set forth in the theorems.

Theorem XI: If $\sum_{n=1}^{\infty} \frac{1}{h_n(x)}$ diverges then,

(12) $\Omega_1(x) = \sum\limits_{n=1}^{\infty} \dfrac{2Q_n}{(x + h_2(x))(x + h_3(x))\ldots(x + h_n(x))}$

where

$2Q_n = c_n + c_{n-1}(h_n(x) - h_1(x)) + c_{n-2}(h_n(x) - h_1(x))(h_{n-1}(x) - h_1(x)) +$

$\ldots + c_1(h_n(x) - h_1(x))(h_{n-1}(x) - h_1(x))\ldots(h_2(x) - h_1(x)).$

Proof: Let $\dfrac{1}{x + h_1(x)} = \dfrac{1}{x + h_2(x)} + \varepsilon_1$, where

$\varepsilon_1 = \dfrac{h_2(x) - h_1(x)}{(x + h_1(x))(x + h_2(x))}$;

$\varepsilon_1 = \dfrac{h_2(x) - h_1(x)}{(x + h_2(x))(x + h_3(x))} + \varepsilon_2$, where

$\varepsilon_2 = \dfrac{(h_2(x) - h_1(x))(h_3(x) - h_1(x))}{(x + h_1(x))(x + h_2(x))(x + h_3(x))}$.

In general with the same notation we have

$\varepsilon_n = \dfrac{(h_2(x) - h_1(x))(h_3(x) - h_1(x))\ldots(h_{n+1}(x) - h_1(x))}{(x + h_1(x))(x + h_2(x))\ldots(x + h_{n+1}(x))}$

From this

$\varepsilon_n = \dfrac{1}{x + h_1(x)} \dfrac{1}{\left(1 + \dfrac{x+h_1(x)}{h_2(x)-h_1(x)}\right)\left(1 + \dfrac{x+h_1(x)}{h_3(x)-h_1(x)}\right)\ldots\left(1 + \dfrac{x+h_1(x)}{h_{n+1}(x)-h_1(x)}\right)}$

This approaches zero since $\sum\limits_{1}^{\infty} \dfrac{1}{h_n(x)}$ diverges. Consequently we have

$\dfrac{c_1}{x + h_1(x)} = \dfrac{c_1}{x + h_2(x)} + \dfrac{c_1(h_2(x) - h_1(x))}{(x + h_2(x))(x + h_3(x))} + \ldots\ldots +$

$$+ \frac{c_1(h_2(x) - h_1(x))(h_3(x) - h_1(x))\ldots(h_n(x) - h_1(x))}{(x + h_2(x))(x + h_3(x))\ldots(x + h_{n+1}(x))} + \ldots .$$

In precisely the same way we find

$$\frac{c_2}{(x + h_1(x))(x + h_2(x))} = \frac{c_2}{(x + h_2(x))(x + h_3(x))} + \frac{c_2(h_3(x) - h_1(x))}{(x+h_2(x))(x+h_3(x))(x+h_4(x))} +$$

$$\ldots + \frac{c_2(h_3(x) - h_1(x))(h_4(x) - h_1(x))\ldots(h_{n-1}(x) - h_1(x))}{(x + h_2(x))(x + h_3(x))\ldots(x + h_n(x))} + \ldots$$

Proceeding in this manner

$$\frac{c_j}{(x + h_1(x))(x + h_2(x))\ldots(x + h_j(x))} = \frac{c_j}{(x + h_2(x))(x + h_3(x))\ldots(x + h_{j+1}(x))}$$

$$+ \frac{c_j(h_{j+1}(x) - h_1(x))}{(x + h_2(x))(x + h_3(x))\ldots(x + h_{j+2}(x))} + \frac{c_j(h_{j+1}(x) - h_1(x))(h_{j+2}(x)-h_1(x))}{(x+h_2(x))(x+h_3(x))\ldots(x+h_{j+3}(x))} +$$

$$+ \ldots .$$

These series considered together form a double series which summed by rows produces $\Omega_1(x)$. This double series is absolutely convergent. To see this do to series (11) for $\Omega_2(x)$ precisely what we have just done to series (10) for $\Omega_1(x)$. The resulting double series is of all positive terms and can be summed to $\Omega_2(z)$ by rows. It consequently is convergent. This double series is a majorant series for the series just obtained for $\Omega_1(x)$ which consequently is absolutely convergent and can be summed by columns. We got the result of the theorem.

We repeat the process that we have just gone through with on series (12). We find

$$\Omega_1(x) = \sum_{n=1}^{\infty} {}_3\varphi_n / (x + h_3(x))(x + h_4(x))\ldots(x + h_{n+2}(x))$$

Here ${}_3\varphi_n = {}_2\varphi_n + {}_2\varphi_{n-1}(h_{n+1}(x)-h_2(x)) + {}_2\varphi_{n-2}(h_{n+1}(x)-h_2(x))(h_n(x)-h_2(x))$

$+\ldots+ {}_2\varphi_1(h_{n+1}(x)-h_2(x))(h_n(x)-b_2(x))\ldots(h_3(x)-h_2(x))$

Proceeding in this manner we have the following theorem.

Theorem XII:

$$\Omega_1(x) = \sum_{n=1}^{\infty} {}_k\varphi_n / (x + h_k(x))(x + h_{k+1}(x))\ldots(x + h_{n+k-1}(x)),$$

__where__ ${}_k\varphi_n = {}_{k-1}\varphi_n + {}_{k-1}\varphi_{n-1}(h_{n+k-2}(x) - b_{k-1}(x)) +\ldots$

$+ {}_{k-1}\varphi_1(h_{n+k-2}(x) - h_{k-1}(x))\ldots(h_k(x) - h_{k-1}(x)).$

4. Step Down Theorem.

Theorem XIII: If

$$\sum(x) = \frac{a_1}{x + h_p(x)} + \frac{a_2}{(x+h_p(x)(x+h_{p+1}(x))} +\ldots+ \frac{a_n}{(x+h_p(x)(x+h_{p+1}(x))\ldots(x+h_{p+n-1}(x))}$$

$+\ldots$ __then__

$$\sum(x) = \frac{a_1}{x + h_{p-1}(x)} + \frac{a_1(h_{p-1}(x) - h_p(x)) + a_2}{(x + h_{p-1}(x))(x + h_p(x))} +\ldots+$$

$$\frac{a_{n-1}(h_{p-1}(x) - h_{p+n-1}(x)) + a_n}{(x + h_{p-1}(x))(x + h_p(x))\ldots(x + h_{p+n-2}(x))} +\ldots$$

Proof:

$$\frac{1}{x + h_p(x)} = \frac{1}{x + h_{p-1}(x)} + \varepsilon \qquad \varepsilon = \frac{h_{p-1}(x) - h_p(x)}{(x + h_p(x))(x + h_{p-1}(x))}$$

$$\frac{1}{(x + h_p(x))} = \frac{1}{x + h_{p-1}(x)} + \frac{h_{p-1}(x) - h_p(x)}{(x + h_p(x))(x + h_{p-1}(x))}$$

$$\frac{1}{(x + h_p(x))(x + h_{p+1}(x))} = \frac{1}{x + h_{p-1}(x))(x + h_p(x))} + \frac{h_{p-1}(x) - h_{p+1}(x)}{(x + h_{p-1}(x))(x + h_p(x))(x + h_{p+1}(x))}$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

This step down process can be continued as many times as desired.

Theorem XIV: If $\rho(x)$ is any function we can write

$$\Omega_1(s) = \sum_{j=1}^{\infty} \overline{c}_j \frac{A(h, \rho)}{(x + \rho(x) + h_1(x))(x + \rho(x) + h_2(x))\ldots(x + \rho(x) + h_j(x))}$$

Proof: Simply apply the ε process of theorem XI.

5. Multiplication Theorem.

Theorem XV: **Let**

$$\Omega(x) = \sum_{n=0}^{\infty} a_{n+1}/(x + h_1(x))(x + h_2(x))\ldots(x + h_{n+1}(x))$$

and

$$\overline{\Omega}(x) = \sum_{n=0}^{\infty} b_{n+1}/(x + h_1(x))(x + h_2(x))\ldots(x + h_{n+1}(x))$$

then

$$(13) \quad \Omega(x)\overline{\Omega}(x) = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} b_j \; b_{j+1} \; \varphi_n/(x + h_{j+1})\ldots(x + h_{j+n+1})$$

Proof: By theorem XI .

$$\Omega(x) = \sum_{n=0}^{\infty} a_{n+1} \frac{1}{(x + h_1(x))\ldots(x + h_{n+1}(x))}$$

$$\Omega(x) = \sum_{n=0}^{\infty} {}_2\varphi_n \frac{1}{(x + h_2(x))\ldots(x + h_{n+2}(x))}$$

$$\Omega(x) = \sum_{n=0}^{\infty} {}_3\varphi_n \frac{1}{(x + h_3(x))\ldots(x + h_{n+3}(x))}$$

. . . .

$$\Omega(x) = \sum_{n=j}^{\infty} {}_j\varphi_n \frac{1}{(x + h_j(s))\ldots(x + h_{j+n}(s))}$$

We now multiply $\Omega(x)$ by each term of $\bar{\Omega}(x)$ as indicated and add

$$\frac{h_1}{x + h_1(x)} \sum_{n=0}^{\infty} {}_2\varphi_n \frac{1}{(x + h_2(x))\ldots(x + h_{n+2}(x))}$$

$$\frac{b_2}{(x + h_1(x)(x + h_2(x)} \sum_{n=0}^{\infty} {}_3\varphi_n \frac{1}{(x + h_3(x))\ldots(x + h_{n+3}(x))}$$

$$\frac{b_3}{(x + h_1(x))(x + h_2(x))(x + h_3(x))} \sum_{n=0}^{\infty} {}_4\varphi_n \frac{1}{(x + h_4(x))\ldots(x + h_{n+4}(x))}$$

. . . .

. . . .

$$\frac{b_j}{(x + h_1(x))\ldots(x + h_j(x))} \sum_{n=0}^{\infty} {}_{j+1}\varphi_n \frac{1}{(x + h_{j+1}(x))\ldots(x + h_{n+j+1}(x))}$$

The double series which appears here can be summed by columns in as much as each row is absolutely convergent and it can be summed by rows. This follows from the fact that each of the series for $\Omega(z)$ converges absolutely. It is interesting to note that z occurs in the numerators of the series (13) only through h. Consequently if h is a constant we can write

$$\overline{\Omega}(x)\, \Omega(x) = \sum_{n=1}^{\infty} \frac{A_n}{(x + h_1)(x + h_2)\ldots(x + h_{n+1})}$$

the A's are constant. If $h_n = n$ we have ordinary factorial series, as is already known.

We are careful to note that the product converges uniformly over the common region of convergence of the two factors.

REFERENCES

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